MPS115/MPS116: HOMEWORK 3

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1. The square-root of 2

We are going to investigate a solution of the equation

$$x^2 = 2. \tag{1}$$

Definition 1.1. The positive solution to equation (1) is denoted $\sqrt{2}$.

First we will prove a useful lemma.

Lemma 1.2. *Any non-zero rational number can be written in the form $\frac{a}{b}$, where a and b are coprime integers.*

Proof. Let q be any non-zero rational number. Then, by definition, $q = \frac{c}{d}$ for some integers c and d, possibly sharing factors. Let h denote the highest common factor of c and d, and write $a = c/h$ and $b = d/h$. Then a and b are coprime integers and $q = \frac{c}{d} = \frac{c}{d} \cdot \frac{1/h}{1/h} = \frac{a}{b}$, as required. □

Now we can use the lemma to prove the following theorem.

Theorem 1.3. *The real number $\sqrt{2}$ is irrational.*

Proof. We will argue by contradiction. Suppose that $\sqrt{2}$ is not irrational; that is to say, suppose that $\sqrt{2}$ is rational. Then, by Lemma 1.2, $\sqrt{2} = \frac{a}{b}$, where a and b share no common factors. Thus, squaring, we find that $2b^2 = a^2$. It follows that a^2 is even, and hence so is a. Write $a = 2m$ for some $m \in \mathbb{Z}$. Then $2b^2 = (2m)^2 = 4m^2$, so $b^2 = 2m^2$. As before, this means that b is even. But then a and b share a factor of 2, which is a contradiction. Hence $\sqrt{2}$ is irrational. □