MAS116/7: HOMEWORK 5

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Problem 1. Let f_n denote the nth *Fibonacci number*. Thus $f_1 = 1$, $f_2 = 1$ and $f_{n+2} = f_{n+1} + f_n$ for every $n \in \mathbb{N}$. Use induction to show that

$$f_1^2 + f_2^2 + \cdots + f_n^2 = f_n f_{n+1}$$

for every natural number $n \in \mathbb{N}$.

Solution. First, for $n \in \mathbb{N}$ let $P(n)$ be the statement

$$f_1^2 + f_2^2 + \cdots + f_n^2 = f_n f_{n+1}.$$

We need to show $P(n)$ is true for every $n \in \mathbb{N}$. We proceed by induction.

For the base case we need to show $P(1)$ is true, that is we need to show that $1^2 = 1 \times 1$ is true, but that is clearly the case. So the base case holds.

For the inductive step, we assume that $P(k)$ is true for some $k \in \mathbb{N}$ and need to show that $P(k+1)$ is true. Using the inductive hypothesis and the definition of f_{k+2}, we have

$$\begin{aligned} f_1^2 + f_2^2 + \cdots + f_{k+1}^2 &= (f_1^2 + f_2^2 + \cdots + f_k^2) + f_{k+1}^2 \\ &= f_k f_{k+1} + f_{k+1}^2 \\ &= f_{k+1}(f_k + f_{k+1}) \\ &= f_{k+1} f_{k+2}. \end{aligned}$$

Thus $P(k+1)$ is true.

Hence, by induction, $P(n)$ is true for all $n \in \mathbb{N}$. □