MAS116/117: HOMEWORK 4

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1. Lagrange Interpolation

Lagrange interpolation is a method used to fit smooth polynomial curves to sets of points in the plane. Suppose we have n distinct points in the plane, where n is an integer greater than 1, with no two x-coordinates equal. Then there is a polynomial $f(x)$ known as the *Lagrange polynomial* of degree at most $n-1$ which passes through the points. In some sense the Lagrange polynomial is the simplest smooth curve that fits the points.

1.1. **The quadratic case.**

Suppose we have three points as described above, (x_0, y_0), (x_1, y_1) and (x_2, y_2), say. Set the following:

$$\begin{aligned} p_0(x) &:= \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} y_0; \\ p_1(x) &:= \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} y_1; \\ p_2(x) &:= \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} y_2. \end{aligned}$$

Define the Lagrange polynomial for the chosen points by $f(x) := p_0(x) + p_1(x) + p_2(x)$.

Notice that

$$p_0(x_0) = \frac{(x_0-x_1)(x_0-x_2)}{(x_0-x_1)(x_0-x_2)} y_0 = y_0,$$

whilst $p_0(x_1) = p_0(x_2) = 0$. Similarly $p_1(x_1) = y_1$, with $p_1(x_0) = 0$ and $p_1(x_2) = 0$; and also $p_2(x_2) = y_2$, with $p_2(x_0) = 0$ and $p_2(x_1) = 0$. It follows that $f(x_0) = y_0$, $f(x_1) = y_1$ and $f(x_2) = y_2$. Further, each of p_0, p_1 and p_2 have degree two. Hence $f(x)$ is a polynomial of degree at most two which passes through (x_0, y_0), (x_1, y_1) and (x_2, y_2); that is, $f(x)$ interpolates our three chosen points.

Note that it is possible that $f(x)$ has degree one or zero. For the points $(0, 0)$, $(1, 1)$ and $(2, 2)$ we find that $f(x) = x$ and so has degree 1, whereas for the points $(0, 0)$, $(1, 0)$ and $(2, 0)$ we find that $f(x) = 0$ and so has degre 0.

Looking at the interpolation for the points $(0, 0)$, $(\frac{\pi}{4}, \frac{1}{\sqrt{2}})$ and $(\frac{3\pi}{4}, \frac{1}{\sqrt{2}})$ which all lie on the curve $y = \sin x$, we end up with $f(x) = \frac{16}{3\pi^2\sqrt{2}} x(\pi - x)$. This is pictured in Figure 1.

1.2. **The general case.**

It is easy to see how this theory generalises. Given n points, (x_i, y_i) for $0 \le i < n$, as described in the introduction, construct

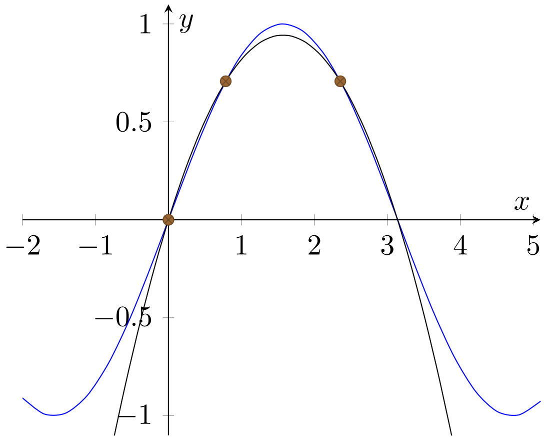

FIGURE 1. The graphs of $y = \frac{16}{3\pi^2\sqrt{2}}x(\pi - x)$ and $y = \sin x$

polynomials $p_i(x)$ for $0 \leq i < n$ by

$$p_i(x) := \prod_{j \neq i} \frac{(x - x_j)}{(x_i - x_j)} y_i.$$

Then, defining $f(x) := \sum_{i=0}^{n-1} p_i(x)$, we have created an interpolating polynomial of degree at most $n-1$. For details, see the Wikipedia page [1].

Figure 2 illustrates the case where $n = 4$, with a cubic curve interpolating the points $(-2, -4)$, $(-1, 1)$, $(0, 2)$ and $(1, 2)$. Here, the Lagrange polynomial turns out to be $f(x) = \frac{1}{2}(x^3 - x^2 + 4)$.

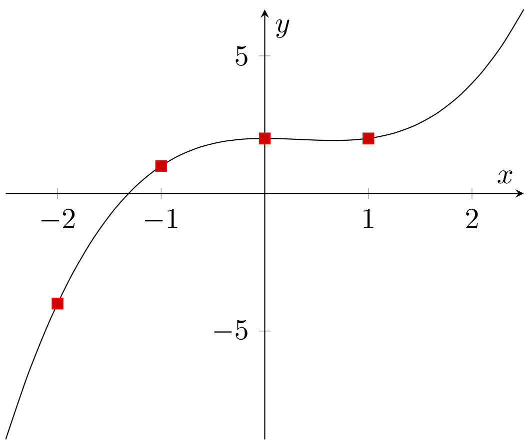

FIGURE 2. The graph of $y = \frac{1}{2}(x^3 - x^2 + 4)$, viewed as a Lagrange polynomial

REFERENCES

[1] Wikipedia contributors, *Lagrange polynomial*, Wikipedia, The Free Encyclopedia. Visited 26 October 2012, updated 25 October 2012, http://en.wikipedia.org/wiki/Lagrange_polynomial.