4. The Karush-Kuhn-Tucker (KKT) conditions [1]

We now introduce the Karush-Kuhn-Tucker (KKT) conditions, which are central in optimisation theory. We will see the power of these conditions later.

We need to start with a few related concepts. These include strong duality and complementary slackness.

4.1. **Strong duality.**

For primal and dual functions as above, we have seen in Proposition 3 that weak duality holds: $v_{\min} \geq f_{\max}$. In the last two examples we saw that equality actually held. This nice special case gets a name.

Definition 4.1.1 (Strong duality). We say that **strong duality** holds if

$$v_{\min} = f_{\max}, \quad \text{i.e.,} \quad v(y^*) = f(x^*),$$

that is, if the optimal costs for the primal and dual problems are the same.

There are problems that don't satisfy strong duality.

Exercise 4.1.2. Consider the following primal problem.

Maximize $f(x_1, x_2) := -e^{-x_1}$ such that $\dfrac{x_1^2}{e^{x_2}} \leq 0$ for $(x_1, x_2) \in \mathbb{R}^2$.

Show that the dual function is given by $v(y) = 0$ for $y \geq 0$ and that

$$v_{\min} = 0 > -1 = f_{\max}.$$

This is a rather contrived example. The problems we will consider will satisfy strong duality. There are easily applicable criteria, "constraint qualifications", to ascertain strong duality of a problem without solving it. However, we will not go into constraint qualifications.

4.2. **Complementary slackness.**

Here we see a fundamental theorem for this section relating primal and dual solutions.

Theorem 4.2.1 (Complementary slackness). *For primal and dual problems as above, with x^* and y^* being the respective optimal solutions, if strong duality holds then the complementary slackness conditions are satisfied:*

$$y_i^* g_i(x^*) = 0 \qquad \textit{for } i = 1, \ldots, m.$$

Note from the proof that if strong duality holds then

$$v(y^*) = L(x^*, y^*) = f(x^*).$$